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THE STRESS FIELDS OF CONTINUOUS DISTRIBUTION OF DISLOCATIONS AND OF THEIR MOVEMENT IN A POLYCRYSTALLINE AGGREGATE†

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Abstract—The elastic fields due to the simultaneous influence of body force, surface force, and the continuous distribution of dislocations are derived. The derivation due to the last part is primarily based on the geometry of dislocations distribution. The obtained results are extended to the elastic–plastic deformation through the concept of dislocations movement. The resultant expressions show that the term associated with the plastic distortional gradient $-C_{ijkl}\beta^{p}_{lk,j}$ and the term associated with the plastic strain $C_{ijkl}\beta^{p}_{lk}n_j$ are equivalent to the body force and the surface force respectively. This result agrees with Lin's "equivalent body force" and "equivalent surface force" obtained in phenomenological plasticity.

1. INTRODUCTION

The main purpose of this work is to derive the stress fields introduced by the continuous distribution of dislocation, and by their movement. The obtained results will apply to both elastic and plastic deformations of single crystals as well as polycrystalline aggregates. The derivation is based on the dislocation geometry. Thus it takes into account the non-uniform behavior of the distribution of dislocations, which occurs in the plastic deformation of metals. Unlike most of the commonly used models in metal plasticity, such as Taylor's constant strain model[1], Batdorf and Budiansky's constant stress model[2], and the self-consistant models of Kröner[3], Budiansky and Wu[4], and Hill[5], the present analysis will provide the exact solution for the calculation of the elastic–plastic deformation of polycrystals. The expressions derived here will also serve to verify Lin's important results of "equivalent body force" and "equivalent surface force"[6–8] which were obtained exclusively from the fundamental equations of continuum plasticity.

In the study of the deformation of metals which is attributed to dislocations, the distribution of dislocations has been considered both as "discrete" and as "continuous." In the former approach, the displacement field caused by a dislocation line in an isotropic medium was first derived by Burgers[9]. Subsequently the stress field introduced by a dislocation line was also obtained by Peach and Koehler[10]. Following this concept various deformation fields due to dislocations with simple geometries, such as cylindrical dislocations, helical dislocations, etc. were discussed by several other investigators[11]. This approach, though it has yielded some interesting results, has been unable to provide significant contributions to continuum plasticity due to its limitation of "discreteness."

Since dislocations exist in crystals with very high density[12], it was found more useful to discuss the deformation fields with the concept of continuous distribution of dislocations. This concept was first applied by Peierls[13] to study the stress field due to an edge dislocation in a periodic structure. But it was Nye[14] who presented this concept more systematically by introducing the dislocation density tensor α_{ij} with

$$B_i = \alpha_{ji}n_j \tag{1}$$

where B_i is the total Burgers vector of the dislocation lines passing through a unit area with unit

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normal n_j. With this definition Bilby[15] and Kröner[16] established the relationships between α_{ij} and the elastic distortional tensor β^e_{ij}, and the plastic distortional tensor β^p_{ij} respectively as

$$\alpha_{ij} = \varepsilon_{ikl}\beta^e_{lj,k} \quad (2)$$

and

$$\alpha_{ij} = -\varepsilon_{ikl}\beta^p_{lj,k} \quad (3)$$

where ε_{ijk} is the unit permutation tensor. In addition to this, Kröner[16] also introduced into the dislocation theory the dislocation movement tensor N_{ijk} as

$$N_{ijk} = nt_i\nu_jb_k \quad (4)$$

with t_i representing the displacement vector of the dislocation with line direction ν_j and Burgers vector b_k, and n the number of such dislocations. While there exist other sets of dislocations at the point of consideration, the total N_{ijk} is obtained by summing up the values of $nt_i\nu_jb_k$ for all sets.

A more systematic study of the deformation fields based on the concept of continuous distribution of dislocations has been carried out by several authors. Among them Kroner[16] elegantly developed a continuum theory of dislocations and internal stress. This theory has greatly simplified the mathematical complexities in this field. Later Indembom[17] extended Kröner's theory of internal stress to study the strain field caused by the continuous distribution of dislocations and obtained for it an integral representation. Mura[18–21] has also greatly contributed to this area in a number of his publications on the static and dynamic dislocations. His analytical work on the periodic distribution of dislocations[19] practically covers all the possible dislocation distributions, since an arbitrary distribution function can always be expanded with sine–cosine terms by means of Fourier series or integrals. In addition Kroupa's concept of the dislocation loop density[22] has even more realistically taken the dislocation geometry into consideration. Publications[16–22] have indeed brought the study of this subject to a new era.

The present investigation could be considered as an extension of the previously cited papers. In order to place the dislocation field in a proper perspective in metal plasticity, the deformation of the solid is here assumed to be under the simultaneous influences of body force, surface force, and the continuous distribution of dislocations. In the first part the elastic deformation will be considered, under which the dislocations remain stationary. In the second part the obtained results will be extended to the condition of elastic–plastic deformation through the concept of dislocations movement.

2. ELASTIC DEFORMATION

When dislocations exist in the crystals, they cause internal distortions around their neighborhoods. By neglecting the singularity at the dislocation cores and the nonlinearity introduced by the dislocation curvatures, these internal distortions can be generally considered to obey the theory of linear elasticity. This assumption will be adopted here. To derive the displacement and stress fields in the continuum under the body force f_i, the boundary surface force F_i and the continuous distribution of dislocations, the following elastic constitutive relation and equation of equilibrium are of importance:

$$\sigma_{ij} = C_{ijkl}u_{k,l} \quad (5)$$

$$\sigma_{ij,j} + f_i = 0 \quad (6)$$

and the boundary condition

$$\sigma_{ij}n_j = F_i \quad \text{on } B \quad (7)$$

where σ_{ij}, C_{ijkl}, u_k, n_j are the stress tensor, elastic constants, displacement vector and unit normal vector to the surface boundary B and $_{,j}$ means the partial differential with respect to the x_j-axis (whereas $_{,j'}$ refers to x'_j-axis). It was also found more convenient to employ Green's tensor function $G_{ij}(x_m - x'_m)$, which represents the displacement u_i at a point x_m arising from a unit point force at x'_m in the x_j-direction in a uniform material. It's defining equation, according to Seeger[23], is

$$C_{ijkl}G_{km,lj} + \delta_{im}\delta(r) = 0 \tag{8}$$

where δ_{im} is the Kronecker's delta, and $\delta(r)$ is the three-dimensional Dirac's delta function.

The displacement field due to a dislocation line in terms of Green's tensor function is given in [11,18]. In order to facilitate our later study, we will briefly derive it here again. We start our derivation from the identity:

$$C_{ijkl}u_i(x'_n)G_{km,l'j'}(x_n - x'_n) = C_{ijkl}u_k(x'_n)G_{im,l'j'}(x_n - x'_n) \tag{9}$$

where the symmetric property, $C_{ijkl} = C_{klij}$ of the elastic constants was employed. This identity is integrated over the volume V with respect to x'_n as

$$\int_v C_{ijkl}u_i(x'_n)G_{km,l'j'}(x_n - x'_n)\,\mathrm{d}v' = \int_v C_{ijkl}u_k(x'_n)G_{im,l'j'}(x_n - x'_n)\,\mathrm{d}v'.$$

Using eqn (8) it becomes

$$\int_v u_i(x'_n)\delta_{im}\delta(x_n - x'_n)\,\mathrm{d}v' = -\int_v C_{ijkl}u_k(x'_n)G_{im,l'j'}(x_n - x'_n)\,\mathrm{d}v'$$

or

$$u_m(x_n) = -\int_v C_{ijkl}u_k(x'_n)G_{im,l'j'}(x_n - x'_n)\,\mathrm{d}v'.$$

Applying the divergence theorem,

$$u_m(x_n) = \int_v C_{ijkl}u_{k,l'}G_{im,j'}\,\mathrm{d}v' - \int_s C_{ijkl}u_kG_{im,j'}\,\mathrm{d}s'_l$$

which in turn leads to

$$u_m(x_n) = -\int_v C_{ijkl}u_{k,l'j'}G_{im}\,\mathrm{d}v' + \int_B C_{ijkl}u_{k,l'}G_{im}\,\mathrm{d}s'_j - \int_s C_{ijkl}u_kG_{im,j'}\,\mathrm{d}s'_l.$$

Equations (5) and (6) and the boundary condition on B lead the above expression to

$$u_m(x_n) = \int_v G_{im}(x_n - x'_n)f_i(x'_n)\,\mathrm{d}v' + \int_B G_{im}(x_n - x'_n)F_i(x'_n)\,\mathrm{d}s' - \int_s C_{ijkl}G_{im,j'}(x_n - x'_n)u_k\,\mathrm{d}s'_l. \tag{10}$$

The last term in eqn (10) represents the displacement field caused by the presence of the displacement discontinuity over the surface s. This displacement discontinuity, under the presence of a dislocation, is identified as the Burgers vector b_k, and the surface s is the glide plane of the dislocation. Consequently eqn (10) becomes

$$u_m(x_n) = \int_v G_{im}(x_n - x'_n)f_i(x'_n)\,\mathrm{d}v' + \int_B G_{im}(x_n - x'_n)F_i(x'_n)\,\mathrm{d}s' - \int_s C_{ijkl}G_{im,j'}(x_n - x'_n)b_k\,\mathrm{d}s'_l. \tag{11}$$

The elastic distortion β^e_{nm}, defined by $\beta^e_{nm} = u_{m,n}$ is then given as

$$\beta^e_{mn} = \int_v G_{im,n}(x_n - x'_n)f_i(x'_n)\,\mathrm{d}v' + \int_B G_{im,n}(x_n - x'_n)F_i(x'_n)\,\mathrm{d}s' + \int_s C_{ijkl}G_{im,jn}(x_n - x'_n)b_k\,\mathrm{d}s'_l. \tag{12}$$

Equation (12) gives the elastic distortion field due to the simultaneous influences of body force f_i, surface force F_i, and a dislocation line with its Burgers vector b_k and glide surface s.

With the field eqns (11) and (12) established for a single dislocation line, we will proceed to extend it to the condition of continuous distribution of dislocations. In this process the theory of linear elasticity will be applied. Denote the last term in eqn (12) as I_{nm}

$$I_{nm} = \int_s C_{ijkl} G_{im,jn}(x_n - x_n')b_k \, ds_l'.$$

After some mathematical manipulations (see Appendix 1) this surface integral is transformed to a line integral as

$$I_{nm} = \oint_c \varepsilon_{njp} C_{ijkl} G_{km,l}(x_n - x_n') b_i \, dx_p' \tag{13}$$

where c is the dislocation line. The linear theory of elasticity implies that, while there exists more than one dislocation, the elastic distortion field caused by these dislocations could be obtained by the linear superposition of the distortion fields introduced by these dislocations individually. Under the condition of several dislocations eqn (13) could thus be modified to

$$I_{nm} = \oint_{c_1, c_2, \ldots c_n} \varepsilon_{njp} C_{ijkl} G_{km,l}(x_n - x_n')[\overset{(1)}{b}_i \, d\overset{(1)}{x}_p + \overset{(2)}{b}_i \, d\overset{(2)}{x}_p + \cdots + \overset{(n)}{b}_i \, d\overset{(n)}{x}_p] \tag{14}$$

where $\overset{(k)}{b}_i$ is the Burgers vector of the kth dislocation whose segment at point x_n' is denoted by $d\overset{(k)}{x}_p$, and c_k is the kth dislocation line.

To express eqn (14) in terms of the established terminology of the continuous distribution of dislocations, consider an element ΔA with a unit normal n_i (see Fig. 1). Assume that there are n dislocations passing through this element. The Burgers vector and line direction of the kth dislocations are denoted by $\overset{(k)}{b}_i$ and $\overset{(k)}{\nu}_i$ respectively. The total Burgers vector B_i, due to all the dislocations enclosed by a Burgers circuit which lies on the surface ΔA and forms a unit area, is

$$B_i = \overset{(1)}{b}_i + \overset{(2)}{b}_i + \cdots + \overset{(m)}{b}_i$$

where m is the total number of dislocations passing through this unit area. To express α_{ji} in terms of $\overset{(k)}{b}_i$, this relation is rewritten as

$$B_i = \sum_{k=1}^{m} \frac{\overset{(k)}{b}_i \overset{(k)}{\nu}_j n_j}{\cos \overset{(k)}{\theta}} \tag{15}$$

where $\overset{(k)}{\theta}$ is the angle between n_i and the kth dislocation line $d\overset{(k)}{x}_i$ (Fig. 2). A comparison of eqns (1) and (15) shows that

$$\alpha_{ij} = \sum_{k=1}^{m} \frac{\overset{(k)}{\nu}_i \overset{(k)}{b}_j}{\cos \overset{(k)}{\theta}}. \tag{16}$$

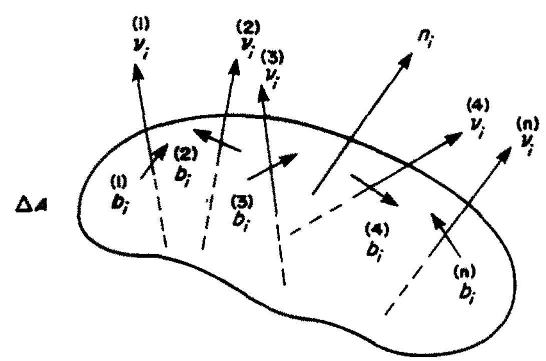

Fig. 1. Dislocations passing through the area element ΔA.

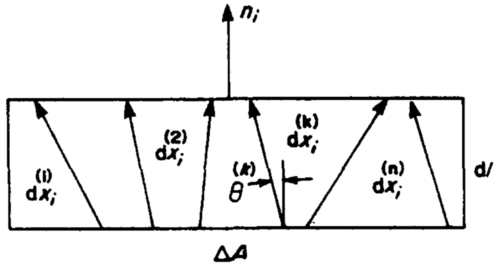

Fig. 2. Dislocation segments inside the volume element dV.

Reconsider eqn (14). The dislocation segments $d\overset{(n)}{x_p}$ are related to the thickness dl of a small volume element dv as shown in Fig. 2. This volume element is made up by the base dA in Fig. 1 and dl in the n_i-direction. The thickness dl is taken to be thin enough so that all dislocations passing through ΔA will emerge also on the top. The segment of the kth dislocation inside this volume is related to dl as

$$d\overset{(k)}{x_i} = \frac{\overset{(k)}{\nu_i}}{\cos \overset{(k)}{\theta}} dl.$$

With this relationship, the quantities inside the bracket of eqn (14) can be rewritten as

$$\overset{(1)}{b_i}\, d\overset{(1)}{x_p} + \overset{(2)}{b_i}\, d\overset{(2)}{x_p} + \cdots + \overset{(n)}{b_i}\, d\overset{(n)}{x_p} = \sum_{k=1}^{n} \frac{\overset{(k)}{\nu_p}\overset{(k)}{b_i}}{\cos \overset{(k)}{\theta}} dl.$$

Since ΔA is not necessary to be a unit area, upon which α_{ij} in eqn (16) was defined, the above quantity can be expressed in terms of the dislocation density tensor as $\alpha_{pi}\Delta A\, dl$, or as $\alpha_{pi}\, dv$. I_{nm} in eqn (14) then becomes

$$I_{nm} = \int_v \varepsilon_{njp} C_{ijkl} G_{km,l}(x_n - x_n')\alpha_{pi}\, dv. \tag{17}$$

According to eqn (12) the elastic distortion β^e_{nm} due to the simultaneous influences of the body force f_i, the boundary surface force F_i, and the continuous distribution of dislocations α_{pi} is then given by

$$\beta^e_{nm} = \int_v G_{im,n}(x_n - x_n')f_i(x_n')\, dv' + \int_B G_{im,n}(x_n - x_n')F_i(x_n')\, ds' + \int_v \varepsilon_{njp} C_{ijkl} G_{km,l}(x_n - x_n')\alpha_{pi}\, dv'. \tag{18}$$

The correspondent stress field is given through the constitutive relation (5) as

$$\sigma_{ij} = \int_v C_{ijmn} G_{rm,n}(x_n - x_n')f_r(x_n')\, dv' + \int_B C_{ijmn} G_{rm,n}(x_n - x_n')F_r(x_n')\, ds' + \int_v \varepsilon_{nsp} C_{ijmn} C_{rskl} G_{km,l}(x_n - x_n')\alpha_{pr}(x_n')\, dv'. \tag{19}$$

It should be pointed out here that a similar expression to the last term of eqn (18) was also found by Indenbom[17] who started from Kröner's concept of internal stress, as well as by DeWit[28]. It is hoped that the present derivation based on the consideration of dislocations geometry would give a clearer picture of the physical meaning of this quantity. On the other hand, Mura[18, 20] also obtained an identical expression for this term. The applicability of his result, however, is inherently limited to the geometrical patterns of parallel dislocations due to his assumption $\alpha_{hi}\, dD = nb_i\, dl_h\, d\Sigma$. In this assumption, the volume element dD, according to Mura, is formed by the dislocation line segment vector dl_h and an area base dΣ normal to dl_h. Since the volume element dD is unique only when all dislocations exist in parallel, the applicability of his result is thus so limited. As dislocations exist in crystals with random orientations, a derivation applicable to arbitrary dislocations distribution as shown here seems more desirable.

In the next section results of eqns (18) and (19) will be extended to the elastic–plastic deformation through the concept of dislocations movement.

3. ELASTIC-PLASTIC DEFORMATION

When a solid is under the simultaneous influences of body force, surface force, and the continuous distribution of dislocations, the stress field σ_{kl} throughout it is given by eqn (19). According to Peach and Koehler[10] the "force" acting on the dislocation is given by $-\varepsilon_{ijk}\sigma_{kl}b_l\nu_j$ for a unit dislocation length. Since a dislocation in a crystal moves only on its glide plane, its "effective" acting force is the resolved component of the acting force on this plane. Let this geometrically permissible movement direction be t_i. The effective acting force is then given by $-\varepsilon_{ijk}\sigma_{kl}b_l\nu_jt_i$ [24], which is here denoted by f_e.

The motion of a dislocation is governed by its effective acting force. Consider a dislocation segment pinned at its two ends. The dislocation line tends to bulge out under the influence of an applied force. The curvature of the bulged dislocation increases with increasing f_e [25]. Though the movement of dislocations in general contributes to plastic strain, the contribution due to this "bulging out" process is nevertheless physically insignificant. As the effective acting force is increased to a critical value f_c, dislocation movement apart from the bulging out process takes place, and this marks the onset of plastic deformation. Most dislocations exist in crystals in their stable configurations. Though there also exist some unstable dislocations with higher potential energy, their number is relatively low and their contribution to the plastic strain is again negligible. It thus could be concluded that before the f_c value of the stable dislocations is reached, no significant plastic strain is generated and the dislocations could be generally assumed to remain stationary. When the effective acting force on the stable dislocations attains the f_c value, dislocations move and plastic strain is produced. The corresponding surface force which provides the f_c value is usually called the "yield point" of the considered material in phenomenological plasticity. When the surface force is below the yield point, the deformation is elastic. While it exceeds this point, the deformation becomes elastic–plastic†.

In the foregoing section we considered elastic deformation under which dislocations remain stationary. In this section the applied surface force is assumed to exceed the yield point; the deformation is elastic–plastic and the dislocations undergo some movement.

Consider an element of a constituent single crystal of polycrystalline aggregate (or just of a single crystal) with elastic–plastic deformation. After the dislocation movement, plastic distortion is generated. The total distortion $u_{i,j}$ of the element can be decomposed into the elastic distortion β^e_{ji} and the plastic distortion β^p_{ji}, as

$$u_{i,j} = \beta^e_{ji} + \beta^p_{ji}. \tag{20}$$

At the end of dislocations movement, dislocations become stationary again. The elastic distortion is again introduced by the body force f_i, the surface force F_i, and the dislocation density tensor α_{ij}, which, under the present circumstance, is the density after the occurrence of the dislocation movement. β^e_{nm} is again representable by eqn (18). To express this representation in terms of plastic distortion β^p_{ji}, Kröner's relation (3) is recalled

$$\alpha_{pi} = -\varepsilon_{pqt}\beta^p_{ti,q}.$$

Substituting this relation in eqn (18), β^e_{nm} becomes

$$\beta^e_{nm} = \int_v G_{im,n}(x_n - x'_n)f_i(x'_n)\,\mathrm{d}v' + \int_B G_{im,n}(x_n - x'_n)F_i(x'_n)\,\mathrm{d}s' + \int_v \varepsilon_{njp}\varepsilon_{qpt}C_{ijkl}G_{km,l}(x_n - x'_n)\beta^p_{ti,q'}\,\mathrm{d}v'. \tag{21}$$

†See [24] for a more detailed discussion on the yield criterion based on dislocation mechanics.

The last integral of eqn (21), again denoted by I_{nm} can be further simplified as follows.

$$I_{nm} = \int_v \varepsilon_{njp}\varepsilon_{qpt}C_{ijkl}G_{km,l}(x_n - x_n')\beta^p_{ti,q'}\,dv'$$

$$= \int (\delta_{nq}\delta_{jt} - \delta_{nt}\delta_{jq})C_{ijkl}G_{km,l'}\beta^p_{ti,q'}\,dv'$$

$$= \int_v C_{ijkl}G_{km,l'}\beta^p_{ji,n'}\,dv' - \int_v C_{ijkl}G_{km,l'}\beta^p_{ni,j'}\,dv'.$$

By the symmetry of $C_{ijkl} = C_{klij}$,

$$I_{nm} = \int_v C_{ijkl}G_{im,j'}\beta^p_{lk,n'}\,dv' - \int_v C_{ijkl}G_{im,j'}\beta^p_{nk,l'}\,dv'.$$

The divergence theorem leads this expression to

$$I_{nm} = -\int_v C_{ijkl}G_{im,j'n'}\beta^p_{lk}\,dv + \int_s C_{ijkl}G_{im,j'}\beta^p_{lk}\,ds_n'$$

$$-\int_s C_{ijkl}G_{im,j'}\beta^p_{nk}\,ds_l' + \int_v C_{ijkl}G_{im,j'l'}\beta^p_{nk}\,dv'$$

$$= \int_v C_{ijkl}G_{im,n'}\beta^p_{lk,j'}\,dv' - \int_s C_{ijkl}G_{im,n'}\beta^p_{lk}\,ds_j'$$

$$+\int_s C_{ijkl}G_{im,j'}\beta^p_{lk}\,ds_n' - \int_s C_{ijkl}G_{im,j'}\beta^p_{nk}\,ds_l'$$

$$+\int_v C_{ijkl}G_{km,j'l'}\beta^p_{nl}\,dv'$$

$$= \int_v C_{ijkl}G_{im,n'}\beta^p_{lk,j'}\,dv' - \int_s C_{ijkl}G_{im,n'}\beta^p_{lk}\,ds_j'$$

$$+\int_s C_{ijkl}G_{im,j'}\varepsilon_{pln}\varepsilon_{prq}\beta^p_{rk}\,ds_q' - \int_v \delta_{im}\delta(x_l - x_l')\beta^p_{nl}\,dv'.$$

Part of the integrand of the third integral, according to Bilby[15], is equal to the surface dislocation density tensor $\bar{\alpha}_{pk}$ at the grain boundaries as

$$\bar{\alpha}_{pk} = \varepsilon_{prq}(\beta^{p(1)}_{rk} - \beta^{p(2)}_{rk})n_q$$

where n_q is the unit normal from grain boundary 1 to boundary 2. I_{nm} is thus simplified to

$$I_{nm} = -\int_v C_{ijkl}G_{im,n}(x_n - x_n')\beta^p_{lk,j'}\,dv' + \int_s C_{ijkl}G_{im,n}(x_n - x_n')\beta^p_{lk}\,ds_j'$$

$$-\int_s C_{ijkl}G_{im,j}\varepsilon_{pln}\bar{\alpha}_{pk}\,ds' - \beta^p_{nm}.$$

By using $C_{ijkl} = C_{klij}$ in the third integrand of I_{nm}, the elastic distortion β^e_{nm} of eqn (21) consequently becomes

$$\begin{aligned}\beta^e_{nm} = &\int_v G_{im,n}[f_i - C_{ijkl}\beta^p_{lk,j'}]\,dv' \\ &+\int_s G_{im,n}[F_i + C_{ijkl}\beta^p_{lk}n_j]\,ds \\ &+\int_s \varepsilon_{nlp}C_{ijkl}G_{km,j}\bar{\alpha}_{pi}\,ds' - \beta^p_{nm}.\end{aligned} \tag{22}$$

By virtue of the deformation decomposition (20), eqn (22) in turn gives the total deformation gradient as

$$\begin{aligned}\frac{\partial u_m}{\partial x_n} = &\int_v G_{im,n}[f_i - C_{ijkl}\beta^p_{lk,j'}]\,dv' \\ &+\int_s G_{im,n}[F_i + C_{ijkl}\beta^p_{lk}n_j]\,ds' + \int_s \varepsilon_{nlp}C_{ijkl}G_{km,j}\bar{\alpha}_{pi}\,ds'.\end{aligned} \tag{23}$$

Upon integration, the total displacement is given by

$$u_m = \int_v G_{im}[f_i - C_{ijkl}\beta^p_{lk,j'}]\,dv' + \int_s G_{im}[F_i + C_{ijkl}\beta^p_{lk}n_j]\,ds' + \int_s \varepsilon_{ljp}C_{ijkl}G_{km}\bar{\alpha}_{pi}\,ds'. \qquad (24)$$

The stress field inside the continuum under this combined elastic–plastic deformation can be determined according to the Hooke's law and eqn (22) as

$$\sigma_{ij} = \int_v C_{ijmn}G_{rm,n}[f_r - C_{rskl}\beta^p_{lk,s'}]\,dv' + \int_s C_{ijmn}G_{rm,n}[F_r + C_{rskl}\beta^p_{lk}n_s]\,ds' + \int_s \varepsilon_{nsp}C_{ijmn}C_{rskl}G_{km,l}\bar{\alpha}_{pr}\,ds' - C_{ijmn}\beta^p_{nm}. \qquad (25)$$

Equations (22)–(25) give the general expressions of the total displacement field, the elastic distortion field and the stress field under the simultaneous influences the body force, the surface force, the surface dislocations at the grain boundaries, and the dislocations movement in the grains which is characterized by the plastic distortional tensor.

When a polycrystal deforms plastically, the deformation fields, due to the dislocations pile-ups against the grain boundaries, are heterogeneous over the entire aggregate. Since the thickness of the grain boundary is only of a few atomic distance, in practical calculation it could be regarded as a wall of zero thickness across which crystal orientation changes from one to another. The heterogeneity of the plastic distortion field indicates that the volume integrals in the foregoing equations should be carried out throughout the entire crystalline aggregate. The fact that crystal orientation changes across the grain boundaries implies that the plastic distortion field is discontinuous over these boundaries. The surface integrals thus have to be carried out over all the grain boundaries. Since there is no surface force applied on the internal grain boundaries, the surface integral corresponding to the surface force exist only on the external ones.

Owing to the nature of their derivation, these equations satisfy both the requirements of equilibrium and compatibility over the grain boundaries. Unlike most theories in metal plasticity mentioned earlier, this system of equations provides an exact solution.

Equations (22)–(25) also show that in the influence of the plastic deformation fields the quantity $-C_{ijkl}\beta^p_{lk,j}$ is equivalent to body force f_i, and the other quantity $C_{ijkl}\beta^p_{lk}n_j$ is equivalent to surface force F_i. These results make it meaningful to call these two terms "equivalent body force" and "equivalent surface force" respectively, as originally proposed by Lin[6]. The deformation fields introduced by these two quantities can be calculated according to the theory of linear elasticity by visualizing them as body force and surface force respectively. For an infinitely isotropic material, the elastic constants C_{ijkl} are given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{jl} + \delta_{il}\delta_{jk})$, and the Green's tensor function is given by Kelvin's solution of unit point force[26]. This substitution reduces the foregoing equations to what Lin employed in his studies on physical plasticity[6–8].

4. CONCLUSION

In this paper the deformation fields under the simultaneous influences of the body force, the surface force and the continuous distribution of dislocations were derived. The derivation attributed to the last factor was based on the consideration of dislocations geometry. The obtained expressions were then extended to the elastic–plastic deformation through the concept of dislocations movement. These results are applicable to arbitrarily distributed dislocations. The final integral representations of the deformation fields show that the terms $-C_{ijkl}\beta^p_{lk,j}$ and $C_{ijkl}\beta^p_{lk}n_j$ are equivalent to body force and surface force respectively. These equivalences are in accordance with Lin's results obtained in phenomenological plasticity.

Since the integral representations of the deformation fields were directly derived from the consideration of dislocations distribution, they are naturally applicable to the inhomogeneous distribution, such as the dislocation pile-ups against the grain boundaries. Both the non-uniform behavior of dislocation pile-ups and the heterogeneity of plastic distortion inside each grain are automatically considered. The conditions of equilibrium and compatibility across the grain boundaries are both satisfied, and the solution is exact.

These equations have paved the way for further study in metal plasticity by means of continuous distribution of dislocations. The elastic distortional energy attributed to the continuous dislocations, for instance, is readily obtainable from eqns (18) and (19). The stress field due to "discrete" dislocations with some simple distribution patterns could also be calculated from eqn (19) by properly defining the corresponding dislocation density tensor. In another study [27] these expressions have served to construct the subsequent yield surfaces of a f.c.c. polycrystalline aggregate.

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APPENDIX 1

Denote

$$I_{nm} = \int_s C_{ijkl} G_{im,jn}(x_n - x'_n) b_k \, ds'_l.$$

The definition of Green's tensor function gives

$$C_{ijkl} G_{km,lj} = -\delta_{im} \delta(x_n - x'_n)$$

and the property of the delta function yields to

$$\int_s b_l \delta_{im} \delta(x_n - x'_n) \, ds'_n = 0$$

where the considered field point x_n in the continuum does not lie on the slip surface s. These two relations together give

$$-\int_s C_{ijkl}G_{km,l'j'}(x_n - x'_n)b_i \, ds'_n = 0.$$

The quantity I_{nm} can therefore be written as

$$I_{nm} = \int_s C_{ijkl}b_i(G_{km,l'n'} \, ds'_j - G_{km,l'j'} \, ds'_n)$$

where the symmetric property of C_{ijkl} was once again used. This quantity can then be represented by

$$I_{nm} = \int_s C_{ijkl}b_i(\delta_{q'j'}\delta_{r'n'} - \delta_{q'n'}\delta_{r'j'})G_{km,l'r'} \, ds'_q$$

or in other words

$$I_{nm} = -\int_s C_{ijkl}b_i\varepsilon_{q'r'p'}\varepsilon_{n'j'p'}G_{km,l'r'} \, ds'_q.$$

This expression now can be transferred to a line integral through Stoke's theorem, which implies that

$$\oint_c T_{p'} \, dx'_p = \int_s \varepsilon_{q'r'p'}T_{p',r'} \, ds'_q.$$

The resultant form after this application is

$$I_{nm} = \oint_c \varepsilon_{njp}C_{ijkl}G_{km,l}(x_n - x'_n)b_i \, dx'_p.$$